

Velocity Estimation Using Adaptive Observers

Marcos A. González Olvera^{*1} and Yu Tang ^{**1}

Department of Control Engineering, Engineering Faculty, National Autonomous University of Mexico, Mexico City. `mangel@verona.fi-p.unam.mx`, `tang@servidor.unam.mx`

Abstract. In this paper we propose a velocity estimator for servo applications based on adaptive observers. Exponential convergence to zero of the estimation error and parameter identification error is established. A numerical example is included to illustrate the results, where a velocity estimation is done even without the direct knowledge of the state equations, only with the knowledge of the output equation.

Keywords: *System Identification, Linear Systems, Adaptive Observer.*

1 Introduction

Linear systems are the basis of many control analysis and synthesis. If a system has a linear state-space representation, several control strategies can be used. In particular for linear time invariant systems, when the states are not available for direct measurement, linear observers can be proposed for estimating these and a linear control strategy may be implemented. The most used and popular one is the Luenberger observer. In this sense, several works have been made and textbooks exist on this subject (see, e.g., [1]). In particular, when the states are not available but the parameters are assumed to be known, effective strategies based on sliding modes have been proposed [2].

However, when the parameters of the linear model are unknown, parametric identification prior to observer and control design must be done. In the case where states and parameters are unknown, and only the output can be measured, the problem turns to be harder to solve. This problem has been studied in the literature under the topic known as adaptive observers, where the strategy focuses on a certain canonic form and the minimal number of parameters that can describe the dynamical behavior of a system. In [3] a succinct description of various adaptive identifiers is described.

In this paper we propose a velocity estimator for servo applications based on adaptive observers [4, 5] for linear systems. For nonlinear systems, a general scheme was presented in [6]. This algorithm tends to be simpler in its implementation with an easier structure than those reported in the literature [7], [8], [3], [9]. Two examples are shown in order to illustrate the effectiveness of the algorithm.

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2 Problem Statement

Let a single-input-single-output system be described by the linear state-space equation

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ y = C\mathbf{x} \end{cases} \quad (1)$$

We assume that C is known, A and B are unknown, the pair (A, C) is observable and y and u are known and measurable. Then, there exists a linear matrix L such that places the eigenvalues of the matrix $A_c = A - LC$ in any location, although its parameters are unknown. In general, we consider that the matrix A_c is Hurwitz.

If this assumptions are fulfilled, then an output injection Ly can be done in the in the form

$$\dot{\mathbf{x}} = (A - LC)\mathbf{x} + LC\mathbf{x} + Bu = A_c\mathbf{x} + Ly + Bu$$

In this sense, the system can be rewritten as

$$\dot{\mathbf{x}} = A_c\mathbf{x} + [I_{n \times n}y \vdots I_{n \times n}u] \begin{bmatrix} L \\ B \end{bmatrix} = A_c\mathbf{x} + \varphi(y, u)\theta \quad (2)$$

where

$$\varphi(y, u) = [I_{n \times n}y \vdots I_{n \times n}u] \quad (3)$$

and

$$L = [l_1 \cdots l_n]^T, B = [b_1 \cdots b_n]^T, \theta = \begin{bmatrix} l_1 \cdots l_n \vdots b_1 \cdots b_n \end{bmatrix}^T = \begin{bmatrix} L \\ B \end{bmatrix}. \quad (4)$$

Now, if the matrices L and B are found by identification, then the dynamics of matrix A can be known; that is, $A = A_c + LC$.

2.1 Persistent excitation condition for $\varphi(y, u)$

Now, in order to assure that in the parametric identification process is done, a persistent excitation condition is needed for $\varphi(y, u)$. This condition is fulfilled if there exists a positive lower bound α such that

$$\int_t^{t+T} \varphi^T \varphi d\tau \geq I\alpha \quad (5)$$

Theorem 1 *The regressor (3) is persistently excited if y and u are nonzero and not linearly dependent in $[t, t + T]$.*

Proof. Without loss of generality, consider $\int_t^{t+T} f(\tau)d\tau$ as $\int f(\tau)d\tau$. Let

$$M = \int_t^{t+T} \varphi^T \varphi d\tau = \int \begin{bmatrix} Iy \\ \vdots \\ Iu \end{bmatrix} [Iy \vdots Iu] d\tau = \int \begin{bmatrix} Iy^2 & \vdots & Iyu \\ \vdots & \ddots & \vdots \\ Iyu & \vdots & Iu^2 \end{bmatrix} d\tau \quad (6)$$

Now define $N = \int I y^2 d\tau$, $P = \int I y u d\tau$ and $Q = \int I u^2 d\tau$. So,

$$M = \begin{bmatrix} N & P \\ P^T & Q \end{bmatrix} \quad (7)$$

We need to check the determinant of M in $[t, t + T]$ in order to evaluate if it loses rank in some point, and so, making $\alpha > 0$ non existent. By its Schur complements, the determinant of M can be evaluated as

$$M = \det(N) \det(Q - P N^{-1} P) \quad (8)$$

For the first factor, as y is nonzero and N is a diagonal matrix,

$$\det(N) = \prod_i \int y^2 d\tau = \left(\int y^2 d\tau \right)^n > 0$$

The second factor can be easily evaluated, as the argument is composed by diagonal matrices. So, we have that

$$\begin{aligned} \det(Q - P N^{-1} P) &= \prod_i \left(\int u^2 d\tau - \frac{\int y u d\tau \int y u d\tau}{\left(\int y^2 d\tau \right)^{-1}} \right) \\ &= \prod_i \left(\frac{\int u^2 d\tau \int y^2 d\tau - \left(\int y u d\tau \right)^2}{\int y^2 d\tau} \right) \end{aligned} \quad (9)$$

By the Cauchy-Schwartz inequality, we know that, if u and y are linearly independent and nonzero in $[t, t + T]$, then

$$\int u^2 d\tau \int y^2 d\tau > \left(\int y u d\tau \right)^2 \quad (10)$$

It follows that $\det(Q - P N^{-1} P) > 0$, and finally $M > 0$, so there exists a positive α such that $M > \alpha I$.

3 Identification based on an Adaptive Observer

To identify L and B , we use the adaptive observer proposed by [4], that makes a dual estimation of states and parameters. The equations that describe this observer are:

$$\dot{\Upsilon} = (A_c - KC)\Upsilon + \varphi(y, u) \quad (11)$$

$$\dot{\hat{\theta}} = \Gamma \Upsilon^T C^T (y - C\hat{x}) \quad (12)$$

$$\dot{\hat{x}} = A_c \hat{x} + \varphi(y, u) \hat{\theta} + (K + \Gamma \Gamma \Upsilon^T C^T)(y - C\hat{x}) \quad (13)$$

where $\mathcal{T}$ is a filtered signal from $\varphi(y, u)$.

This equations are obtained from the following analysis. In first place, consider the linear system defined by (2) to work as a parallel model for (1). The problem turns out into a joint identification problem of both $\mathbf{x}$ and θ . As the pair (A_c, C) is observable, and L is previously designed as an output injection that preserves the observability of the pair (A_c, C) a vector $K \in \mathbb{R}^n$ can be found such that the matrix $A - LC - KC = A_c - KC$ is Hurwitz.

Taking $\varphi = \varphi(y, u)$ and following this, an output injection is considered for the state equation in the form

$$\dot{\mathbf{x}} = (A_c - KC)\mathbf{x} + \varphi\theta + Ky. \quad (14)$$

Following a similar treatment as [5], the states can be decomposed into

$$\mathbf{x} = \mathbf{x}_y + \mathbf{x}_\varphi, \quad (15)$$

where $\mathbf{x}$ is composed by the linear combination of two signals $\mathbf{x}_y, \mathbf{x}_\varphi$,

$$\dot{\mathbf{x}}_y = (A_c - KC)\mathbf{x}_y + Ky, \quad (16)$$

$$\dot{\mathbf{x}}_\varphi = (A_c - KC)\mathbf{x}_\varphi + \varphi\theta. \quad (17)$$

As the dynamics for $\mathbf{x}_y$ are given in a linear form, an estimator for this signal can be obtained by the linear observer

$$\dot{\hat{\mathbf{x}}}_y = (A_c - KC)\hat{\mathbf{x}}_y + Ky, \quad (18)$$

and an estimator for $\mathbf{x}_\varphi$ is proposed to have the form

$$\dot{\hat{\mathbf{x}}}_\varphi = (A_c - KC)\hat{\mathbf{x}}_\varphi + \varphi\hat{\theta} + \mathcal{T}(t)\hat{\theta}, \quad (19)$$

where $\mathcal{T}$ defines a dynamic transformation $\hat{\mathbf{x}}_\varphi = \mathcal{T}(t)\hat{\theta}$. By developing the derivative of $\hat{\mathbf{x}}_\varphi$,

$$\frac{d\hat{\mathbf{x}}_\varphi}{dt} = \dot{\mathcal{T}}\hat{\theta} + \mathcal{T}\dot{\hat{\theta}}, = (A_c - KC)\mathcal{T}\hat{\theta} + \varphi\hat{\theta} + \mathcal{T}\dot{\hat{\theta}}, \quad (20)$$

we have that

$$\dot{\mathcal{T}}\hat{\theta} = (A_c - KC)\mathcal{T}\hat{\theta} + \varphi\hat{\theta}, \quad (21)$$

$$\left(\dot{\mathcal{T}} - (A_c - KC)\mathcal{T} - \varphi\right)\hat{\theta} = 0. \quad (22)$$

This last equation gives

$$\dot{\mathcal{T}} = (A_c - KC)\mathcal{T} + \varphi, \quad (23)$$

which describes a filtered matrix of the regressor φ . Since φ is bounded, so is $\mathcal{T}$ because $A_c - KC$ a Hurwitz matrix. The adaptive algorithm is proposed to be

$$\dot{\hat{\theta}} = \Gamma \mathcal{T}^T C^T (y - C\hat{\mathbf{x}}), \quad (24)$$

where Γ is a positive definite constant gain matrix. The stability and convergence of the proposed observer and the adaptive algorithm are summarized in the following theorem.

Theorem 2 Consider the system (1). Assume that it is observable and so transformable into the equivalent system (2). The following system

$$\dot{\hat{\mathbf{x}}} = A_c \hat{\mathbf{x}} + \varphi(y, u) \hat{\theta} + (K + \Upsilon \Gamma \Upsilon^T C^T) (y - C \hat{\mathbf{x}}), \quad (25)$$

with the adaptive algorithm

$$\dot{\hat{\theta}}_{\mathbf{x}} = \Gamma \Upsilon^T C^T (y - C \hat{\mathbf{x}}), \quad (26)$$

where $\Gamma > 0$, and the gain matrix $K \in \mathbb{R}^n$ is such that $A - KC$ is Hurwitz, and

$$\dot{\Upsilon} = (A_c - KC) \Upsilon + \varphi(y, u), \quad (27)$$

the filtered version of $\varphi(y, u)$, guarantees that all signals are bounded, and the state estimation error $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ and parameter error $\tilde{\theta} = \hat{\theta} - \theta$ are exponentially stable, provided that the regressor matrix (3) is persistently exciting, i.e.,

$$\int_t^{t+T} \varphi^T(\tau) \varphi(\tau) d\tau \geq \alpha I. \quad (28)$$

for some $0 < \alpha$, and $T > 0$, considering $\varphi(\tau) = \varphi(y(\tau), u(\tau))$.

Proof. The smooth function $Ly + Bu$ can be written as $\varphi(y, u) \hat{\theta}$, and then get $Ly + Bu - \varphi(y, u) \hat{\theta} = 0$, if $\theta = \hat{\theta}$. In this sense, the system (1) can be represented as

$$\dot{\mathbf{x}} = A_c \mathbf{x} + \varphi(y, u) \theta, \quad (29)$$

$$y = C \mathbf{x}. \quad (30)$$

Now, taking $\varphi = \varphi(y, u)$ and considering (25) such that $\hat{\mathbf{x}}$ is meant to be an estimate of $\mathbf{x}$, we define the error signal

$$\xi(t) = \tilde{\mathbf{x}} - \Upsilon \tilde{\theta}. \quad (31)$$

where $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$, $\tilde{\theta} = \theta - \hat{\theta}$.

The derivative of the previous equation represents the error signal dynamics, which are

$$\dot{\xi}(t) = (A_c - KC) \xi, \quad (32)$$

$$\dot{\tilde{\theta}} = -\Gamma \Upsilon^T C^T C (\xi + \Upsilon \tilde{\theta}) = -\Gamma \Upsilon^T C^T C \Upsilon \tilde{\theta} - \Gamma \Upsilon^T C^T C \xi.$$

It follows that ξ is exponentially stable, as the matrix $A - KC$ is designed to be Hurwitz. Moreover, Υ must be bounded as is obtained from a stable filtering of φ , that is also bounded.

Now, it can be proven [5, 10] that under a persistent excitation condition (28) the origin of the system

$$\dot{\chi} = -\Gamma \varphi^T \varphi \chi \quad (33)$$

is exponentially stable. It follows that the homogeneous part of (33) is also exponentially stable. So, finally, $\lim_{t \rightarrow \infty} \tilde{\theta} = 0$.

4 Examples

In this section, two examples are given in order to demonstrate the effectiveness of the proposed algorithm. The first example shows how this algorithm can estimate the velocity of a DC motor without direct derivation, even when the parameters of the state equation matrices are unknown and only the output matrix is known. The second example shows how the proposed algorithm can not only estimate the velocity, but also be used for an effective parameter identification.

4.1 Velocity estimation for a DC motor

Consider a 2nd order system for a DC motor, controlled by a voltage signal $u(t)$ in the form $K_v u(t) = \tau(t)$,

$$K_v u(t) = J\ddot{\theta} + b\dot{\theta} + k\theta \quad (34)$$

where θ is the angle of the rotor, J is the rotor inertia, b the viscous friction, K_v the gain constant for the input and $\tau(t)$ the input torque. In an observable canonical form, the model is

$$\begin{aligned} \dot{\mathbf{x}} &= \overbrace{\begin{bmatrix} -\frac{b}{J} & 1 \\ -\frac{k}{J} & 0 \end{bmatrix}}^A \mathbf{x} + \overbrace{\begin{bmatrix} 0 \\ \frac{K_v}{J} \end{bmatrix}}^B u(t) \\ y &= \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^C \mathbf{x} \end{aligned} \quad (35)$$

Consider the parameters $b = 5$, $J = 10$, $k = 5$ and $K_v = 100$. For this case, it is designed an observer for estimating the velocity when only the output measurement is available, and no information on any parameter is available. It is just known that the pair (A, C) is observable, so it can be transformed into an observable canonical form (not necessarily in any canonical form). For this case, the system (34) takes the form

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & -\frac{b}{J} \end{bmatrix}}^{\bar{A}} \bar{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ \frac{K_v}{J} \end{bmatrix}}^{\bar{B}} u \\ y &= \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{\bar{C}} \bar{\mathbf{x}} \end{aligned} \quad (36)$$

where we know that its observable canonical form is (35), and that the relation between $\mathbf{x}$ and $\bar{\mathbf{x}}$ is

$$\bar{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ -\frac{b}{J} & 1 \end{bmatrix} \mathbf{x} \quad (37)$$

and that the angular velocity of the system $\omega = \dot{\theta}$ is $\omega = \bar{x}_2 = -\frac{b}{J}x_1 + x_2$. So, if the observer and identifier can estimate x_2 , then the angular velocity can be also estimated by

$$\hat{\omega} = \hat{x}_2 = -\frac{\hat{b}}{J}\hat{x}_1 + \hat{x}_2 \quad (38)$$

and

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = (A - LC)\mathbf{x} + Bu + Ly = \overbrace{(A - LC)}^{A_d} \mathbf{x} + \varphi(y, u)\theta \quad (39)$$

It is known that $x_1 = \theta$ and $x_2 = \dot{\theta}$. Now, we define the *desired* dynamics for the system by choosing

$$A_d = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix} \quad (40)$$

So, by identifying $\hat{L}$ and $\hat{B}$, the parameter relations $\frac{b}{J}$ and $\frac{k}{J}$ can be estimated by $\frac{b}{J} = -(a_{11} + l_1)$ and $\frac{k}{J} = -(a_{21} + l_2)$. By applying the transformation (38), we can estimate the angular velocity of the system.

In Fig. 1.a it can be seen how the velocity estimation $\hat{x}_2$ tends to the real velocity x_2 , still when the parameters are unknown. For this simulation, the parameter adaptation is stopped at $t = 100[s]$, and then a different input signal is used in order to test the effectiveness of the estimation. In Fig. 1.b it is shown the performance of the velocity estimator after the estimation is suspended and the input changed. It can be seen that the error is practically zero.

4.2 Parametric identification of a 2nd order system

In this example we will show that the proposed algorithm is not only capable of estimate the velocity of an unknown system, but also helps to achieve a parametric identification. The system (34) will be assumed once again to have unknown parameters. For this example, let the *nominal* parameters be $b = 0.1$, $J = 10$, $k = 3$ and $K_V = 2$. For the identifier, the desired dynamics for matrix $A_c = (A - LC)$ are such that the eigenvalues are $\{-1.5, -2.5\}$, and the desired dynamics for the matrix $A_c - LC$ are chosen such that its eigenvalues are located in $\{-10, -12\}$ and $\Gamma = 10^3 I_{2 \times 2}$. The training was done with a random input signal with unitary variance and zero mean, such that changes every 5s. After training, it was obtained

$$\hat{L} = \begin{bmatrix} 3.99 \\ 3.45 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1.0857 \times 10^{-5} \\ 0.2 \end{bmatrix}. \quad (41)$$

From the nominal parameters, it is obtained that $b_1 = 0$ and $b_2 = 0.2 = \frac{K_V}{J}$, $a_{11} = -\frac{b}{J}$, $a_{21} = -\frac{k}{J}$. After training, and solving the previous equations for $\hat{b}$, $\hat{k}$ and $\hat{J}$, assuming only K_V is known, it is obtained that $\hat{J} = 10.0017$, $\hat{b} = 0.09999$ and $\hat{k} = 3.0005$. It is clear that these parameters remain quite close to the real ones. In Fig. 2a it can be seen the evolution of the parameters during training, while in Fig. 2b and 2c a comparison of the parallel response of both systems is compared, with a RMS error of 0.0048.

5 Conclusions

As estimator based on adaptive observer theory for unknown (assumed observable) linear systems was presented. In particular, the application was focused on the velocity

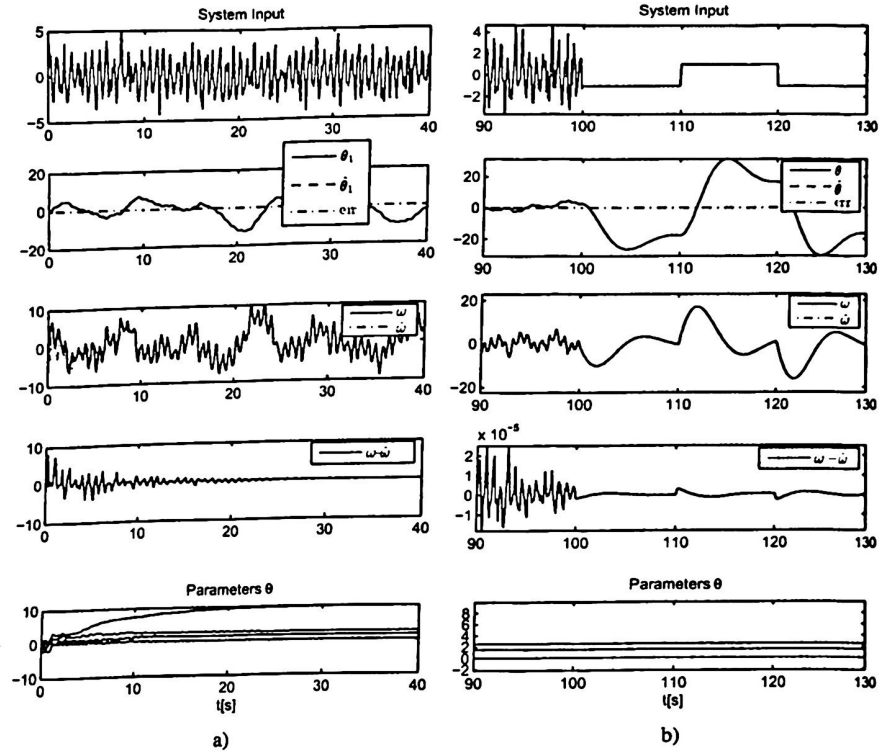


Fig. 1. Simulation results for Example 3. a) During estimation $t \in [0, 40][s]$, b) Estimation stopped at $t = 100[s]$

estimation for 2nd order mechanical systems. Two examples were shown to demonstrate the effectiveness of the method. It was also demonstrated that the method can be employed for parameter identification or velocity estimation, where only the position is assumed measurable and the parameters are unknown. The future work includes the analysis for higher order systems and experimental applications.

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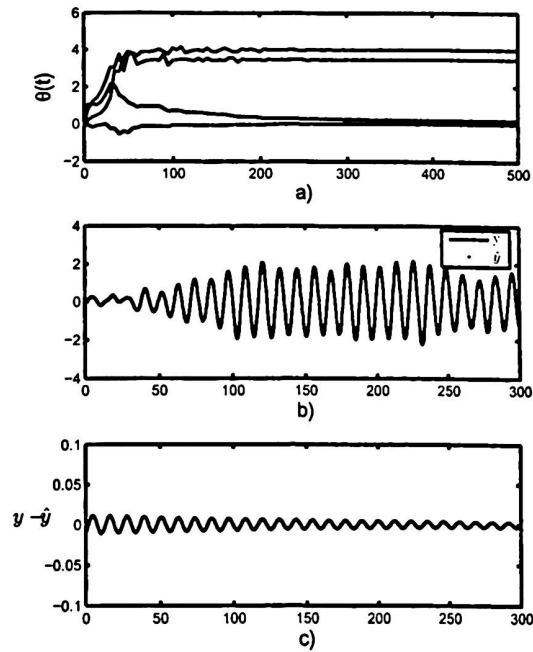


Fig. 2. Simulation results for Example 2

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